

# An analysis of a random algorithm for estimating all the matchings

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Counting the number of all the matchings on a bipartite graph has been transformed into calculating the permanent of a matrix obtained from the extended bipartite graph by Yan Huo, and Rasmussen presents a simple approach (RM) to approximate the permanent, which just yields a critical ratio  $O(n\omega(n))$  for almost all the 0-1 matrices, provided it's a simple promising practical way to compute this #P-complete problem. In this paper, the performance of this method will be shown when it's applied to compute all the matchings based on that transformation. The critical ratio will be proved to be very large with a certain probability, owning an increasing factor larger than any polynomial of  $n$  even in the sense for almost all the 0-1 matrices. Hence, RM fails to work well when counting all the matchings via computing the permanent of the matrix. In other words, we must carefully utilize the known methods of estimating the permanent to count all the matchings through that transformation.

Keywords: matching; permanent; critical ratio; bipartite graph; determinant; Monte-Carlo algorithm; random algorithm; RM; fpras

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## I. INTRODUCTION

Let  $G = (V, E)$  be a bipartite graph, where  $V = V_1 \cup V_2$  is the set of vertices and  $E \subset V_1 \times V_2$  is the set of edges. In the following sections we suppose  $\#V_1 = \#V_2 = n$  if there's no special illustration. A set of edges  $S \subset E$  is called a matching if no two distinct edges  $e_1, e_2 \in S$  contain a common vertex.  $S$  is called a  $k$ -matching if  $\#S = k$ . In special case,  $S$  is called a perfect matching if  $k = n$ . Let  $S_k$  be the set of  $k$ -matching in  $G$  and  $A(G)$  be the set of all the  $k$ -matching,  $k = 0, 1, \dots, n$ . For the convenience of discussion, let  $\#S_0 = 1$ , then the number of all the matchings in  $G$  is  $\#A(G) = \sum_{i=0}^n \#S_i$ .

The permanent of a 0-1  $A = a_{ij}, 1 \leq i, j \leq n$  is defined as

$$Per(A) = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)} \quad (1)$$

where the sum is over all the permutations  $\pi$  of  $[n] = \{1, \dots, n\}$ . It's well known that the permanent of an adjacent matrix of bipartite graph equals the number of its perfect matching. Let  $AM(G)$  denote the number of all the matchings in  $G$ , and  $A$  be adjacent matrix of  $G$ . [8] has proved that

$$AM(G) = \frac{1}{n!} per \begin{pmatrix} A & I_{n \times n} \\ 1_{n \times n} & 1_{n \times n} \end{pmatrix} \quad (2)$$

where  $I_{n \times n}$  is  $n \times n$  unit matrix,  $1_{n \times n}$  denotes  $n \times n$  matrix with all the elements 1. This means in order to count the number of all the matchings of a bipartite graph with  $2n$  vertices we only need to compute the permanent of a  $2n \times 2n$  corresponding matrix transformed from adjacent matrix. The computation of permanent has a long history and was shown to be  $\#P$ -complete in [2]. Thus, in the past 20 years or so, many random algorithms have been developed to approximate the permanent, which can be divided at least four categories[3]: elementary recursive algorithms(the original one is Rasmussen method(RM)) [4]; reductions to determinants [5, 7, 9, 11]; iterative balancing [12]; and Markov chain Monte Carlo [13, 16, 19]. All these methods try to find a fully-polynomial randomized approximation scheme *fpras* for computing the permanent. *fpras* is such a scheme which, when given  $\varepsilon$  and inputs matrix  $A$ , outputs a estimator(usually a unbiased estimator) $Y$  of the permanent such that

$$Pr((1 - \varepsilon)per(A) \leq Y \leq (1 + \varepsilon)per(A)) \geq \frac{3}{4} \quad (3)$$

and runs in polynomial time in  $n$  and  $\varepsilon^{-1}$ , here  $3/4$  may be boosted to  $1 - \delta$  for any desired  $\delta > 0$  by running the algorithm  $O(\log(\delta^{-1}))$  and taking the median of the trials [10]. Then a straightforward application of Chebychev's inequality shows that running the algorithm  $O(\frac{E(Y^2)}{E^2(Y)}\varepsilon^{-2})$  times and taking the mean of the results can make the probability more than  $3/4$  (e.g. running  $4\frac{E(Y^2)}{E^2(Y)}\varepsilon^{-2}$  times). Hence, if the critical ratio  $\frac{E(Y^2)}{E^2(Y)}$  is bounded by a polynomial of inputs  $A$ , we'll get an *fpras* for the permanent of  $A$ . Another modified scheme called *fpras* for almost all inputs means: choose a matrix from  $\mathcal{A}(n, 1/2)$  ( $\mathcal{A}(n, 1/2)$  denotes a probability space of  $n \times n$  0-1 matrices where each entry is chosen to be 1 or 0 with the same probability  $1/2$ ), or equivalently choose a matrix u.a.r. from  $\mathcal{A}(n)$  ( $\mathcal{A}(n)$  represents the set of  $n \times n$  0-1 matrices), and the following

$\Pr(\text{critical ratio of } A \text{ is bounded by a polynomial of the input } A) = 1 - o(1)$  as  $n \rightarrow \infty$  holds. (Note that this is a much weaker requirement than that of an *fpras*). If a proposition  $P$  relating to  $n$  satisfies  $\Pr(P \text{ is true}) = 1 - o(n)$ , we say  $P$  holds **whp** (**whp** is the abbreviation of "with high probability"). Thus, that there is an *fpras* for almost all the matrix means the critical ratio of  $A$  is bounded by a polynomial of the input  $A$  **whp**. A exciting result, that Markov Chain approach led to the first *fpras* for the permanent of any 0-1 matrix (actually of any matrix with nonnegative entry) was shown by [16]. However, its high exponent of polynomial running time makes it difficult to be a practical method to approximate the permanent. RM and reductions to determinants seem to be two practical approaches estimating permanent due to their simply feasibility, and both of them have been proved to be an *fpras* for almost all the 0-1 matrices. besides, [3] promises a good prospect on computing permanent via clifford algebra if some difficulties can be conquered. RM also has developed to be a kind of approaches called sequential importance sampling way, which is widely used in statistical physics, see [14].

In this paper, we'll, by RM, compute the number of all the matchings based on the above transformation and give its performance theoretically, say, an analysis of critical ratio in the sense "for almost all the 0-1 matrix" of that matrix with a special structure. In section II, A new alternative estimator operating directly on the adjacent matrix without any transformation will be presented and proved to be equivalent to approximation performing on the transformed matrix by RM. In section III, a low bound of the critical ratio for almost all the matrices will be presented, which is larger than any polynomial of  $n$  with a certain probability. Hence, RM does not perform well in computing the number of all the matchings

as in computing the number of perfect matching. In section IV we'll propose some analytic results w.r.t. the expectation and variance of the number of all the matchings of a matrix selected u.a.r from  $\mathcal{G}(m, n)$  ( $\mathcal{G}(m, n)$  denotes the set of bipartite graph with  $\#V_1 = \#V_2 = n$  as its vertices and exact  $m$  edges). These results seem likely to contribute to the upper bound of critical ratio for almost all matrices, but the calculations are more arduous and will be left for latter paper.

## II. AN EQUIVALENT ESTIMATOR

All the notations have the same meanings as those in the previous section without special illustration. Let  $A$  an  $n \times n$  0-1 matrix be an adjacent matrix of a bipartite graph  $G = (V, E)$ , ( $V = V_1 \cup V_2$ ). Set  $Y_A$  a random variable. Then RM can be stated as follows:

**inputs:**  $A$  an  $n \times n$  0-1 matrix;

**outputs:**  $Y_A$  the estimator of permanent  $A$ ;

if  $n=0$ ; then

$$Y_A = 1$$

else

$$W = \{j : a_{1j} = 1\}$$

if  $W = \emptyset$  then

$$Y_A = 0$$

else

Choose  $J$  u.a.r. from  $W$

$$Y_A = |W|Y_{1J}$$

$Y_{1j}$  denotes the submatrix obtained from  $A$  by removing the 1st row and the  $j$ th column. Note this heuristic idea comes from the Laplace's expansion. Our following algorithm (for easy discussion, call it AMM) is also inspired by another expansion. we first presents our algorithm for the number of all the matchings, and then give the explanation and proof of equivalence between AMM and RM on the transformed matrix:

**inputs:**  $A$  an  $n \times n$  0-1 adjacent matrix of  $G$ ;

**outputs:**  $Y_A$  the estimator of the number of all the matchings of  $G$ ;

if  $n=0$ ; then

$$Y_A = 1$$

else

$$W = \{j : a_{1j} = 1\} \cup \{0\}$$

Choose  $J$  u.a.r. from  $W$

$$Y_A = |W|Y_{1J}$$

$Y_{10}$  denotes a submatrix of  $A$  by removing the 1st row (of course, it's not necessarily a square matrix). Define a new terminology AM on the matrix. let  $B = \{b_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$  an  $m \times n$  matrix,  $m \leq n$ . let  $AM(\emptyset) = 1$ , by induction on  $m$ .

$$AM(B) := AM(B_{10}) + \sum_{j=1}^n b_{1,j} B_{1j} \quad (4)$$

Then we have the following theorem.

**Theorem 1.** Let  $A$  be an  $n \times n$  adjacent matrix of a bipartite graph  $G$ , Then  $AM(A)$  is the number of all the matchings of  $G$ .

**Proof:** It's easy to check, when  $k \geq 1$ , the number of  $k$ -matching of  $G$  equals

$\sum_{i_1, \dots, i_k} \sum_{\pi} a_{i_1, \pi(i_1)} \cdots a_{i_k, \pi(i_k)}$ , where  $i_1 < i_2 < \dots < i_k$  chosen from  $\{1, 2, \dots, n\}$ ,  $\pi$  denotes the permutation of  $\{i_1, i_2, \dots, i_k\}$ . Thus, the number of all the matchings

is  $\sum_{k=1}^n \sum_{i_1 < \dots < i_k \subseteq \{1, \dots, n\}} \sum_{\pi} a_{i_1, \pi(i_1)} \cdots a_{i_k, \pi(i_k)} + 1$ , where 1 denotes the number of 0-matching.

Note that if the  $AM(A)$  is written in terms of sum of elements of the matrix  $A$ , then it's

clearly to see  $AM(A) = \sum_{k=1}^n \sum_{i_1 < \dots < i_k \subseteq \{1, \dots, n\}} \sum_{\pi} a_{i_1, \pi(i_1)} \cdots a_{i_k, \pi(i_k)} + 1. \square$

**Corollary1.** Let  $A = \{a_{ij} | 1 \leq i, j \leq n\}$  be an  $n \times n$  0-1 matrix and  $Y_A$  is obtained by above AMM. Then  $Y_A$  is unbiased for  $AM(A)$ ,  $E(Y_A) = AM(A)$

**Proof:** We prove for any  $m \times n$  0-1 matrix  $A$ ,  $1 \leq i \leq m, 1 \leq j \leq n$ , which will be widely used in the following proves. AMM is unbiased for  $AM(A)$ . For any fixed  $n$ , by induction on  $m$ ,  $k=0, \forall 1 \leq l \leq n$ ,  $\forall$  a  $k \times l$  0-1 matrix  $A$ , the equation  $E(Y_A) = AM(A)$  is trivial. Now suppose  $\forall k \leq m, k \leq l \leq n$ , a  $k \times l$  0-1 matrix  $A$  has  $E(Y_A) = AM(A)$ . Then when  $k = m$ , let  $|W| = q$ , we have

$$\begin{aligned}
E(Y_A) &= \sum_{j \in W} E(Y_A | J = j) Pr(J = j) \\
&= \sum_{j \in W} E(qY_{A_{1j}} | J = j) q^{-1} \\
&= \sum_{j \in W} E(Y_{A_{1j}}) \\
&= \sum_{j \in W} AM(A_{1j}) \\
&= AM(A).
\end{aligned}$$

□

Another simple corollary can also be obtained. To estimate the number of all the matching in  $G$ , by RM operating on  $B = \begin{pmatrix} A & I_{n \times n} \\ 1_{n \times n} & 1_{n \times n} \end{pmatrix}$  divided by  $n!$  is equivalent to operating on  $A$  by AMM, in precise words, which can be stated as follows.

**Corollary2.** Let  $X_A$  be the output of RM operating on  $A$ ,  $Y_B$  be the output of AMM operating on transformed matrix  $B$  divided by  $n!$ . Then  $X_A$  and  $Y_B$  has the same distribution.

**Proof:** Note that by RM after  $n$ -th step operating on  $B$ ,  $Y_B = S_n * Y_{1_{n \times n}} / n!$ , where  $S_n$  is a number obtained from the first  $n$  steps, and obviously  $Y_{1_{n \times n}} \equiv n!$ . Hence, we have  $Y_B = S_n$ . The same distribution of  $S_n$  and  $X_A$  can be verified step by step. □

**Corollary3.**  $AM(A) = \frac{1}{n!} per \begin{pmatrix} A & I_{n \times n} \\ 1_{n \times n} & 1_{n \times n} \end{pmatrix}$ .

**Proof:** This is a direct deduction of corollary2. Let  $X_A$  be the output of RM operating on  $A$ ,  $Y_B$  be the output of AMM operating on transformed matrix  $B$  divided by  $n!$ .

$$AM(A) = E(X_A) = E(Y_B) = \frac{1}{n!} per \begin{pmatrix} A & I_{n \times n} \\ 1_{n \times n} & 1_{n \times n} \end{pmatrix}$$

□

So in the following section, we'll use AMM to compute all the matchings instead of RM since some methodologies similar to Rasmussen can be utilized. Another small advantage

by AMM is that the critical ratio is smaller than that directly obtained from RM. The critical ratio by RM would be  $(2n)!$ , see Theorem 2.2[4], while the critical ratio by AMM would be  $(n+1)^n$ .

**Theorem2.** Let  $A = \{a_{ij}, 1 \leq i, j \leq n\}$  be an  $n \times n$  adjacent matrix of a bipartite graph  $G$ , and let  $X_A$  be the output of AMM. Then  $\frac{E(X_A)^2}{E(X_A^2)} \leq (n+1)^n$ . Generally, Let  $A$  be an  $m \times n$  0-1 matrix,  $m \leq n$ .  $X_A$  be the output of AMM. Then  $\frac{E(X_A)^2}{E(X_A^2)} \leq (n+1)^m$

**Proof:** Induction on  $m$ , For any fixed  $n$ .  $k = 0, \forall 1 \leq l \leq n$ ,  $\forall$  a  $k \times l$  0-1 matrix  $A$ , the inequation is trivial. In the case  $k = m$ , let  $|W| = q$ , we have

$$\begin{aligned}
E(X_A^2) &= \sum_{j \in W} E(X_A^2 | J = j) \Pr(J = j) \\
&= \sum_{j \in W} E(q^2 X_{A_{1j}}^2 | J = j) q^{-1} \\
&= \sum_{j \in W} E(X_{A_{1j}}^2) q \\
&\leq \sum_{j \in W} E(X_{A_{1j}})^2 (n+1)^{m-1} q \\
&\leq \left( \sum_{j \in W} E(X_{A_{1j}}) \right)^2 (n+1)^{m-1} q \\
&= E(X_A)^2 (n+1)^m
\end{aligned}$$

□

### III. A LOWER BOUND OF CRITICAL RATIO FOR ALMOST ALL THE MATRICES

Rasmussen shows that although the critical ratio of RM is factorial in  $n$ , it does indeed provide an fpras for almost all the matrix. However, the similar result can not be anticipated when computing all the matchings by RM. In fact the critical ratio for almost all the matrix would be more than  $n^{\sqrt{n}/2-1}$  with a certain probability. To prove this, we need to define some new denotations. Since there're two probability spaces, we use the subscript  $\sigma$  denote the calculus w.r.t. the probability space the algorithm lies in, say, coin-tosses, and subscript  $\mathcal{A}$  represent the calculus w.r.t. the space probability the random matrices lie in.  $\mathcal{A}(m, n, p)$  denotes the probability space of all  $m \times n$  0-1 random matrices where each entry is chosen

to be 1 with probability  $p$ , and  $\mathcal{A}(m, n)$  denotes the set of all  $m \times n$  0-1 matrices .

To obtain the mean and variance of the output of AMM on average under probability measure  $Pr_{\mathcal{A}}$ , we need the following lemma.

**Lemma1** Let  $f(m, n)$  defined as  $f(m, n) = a_n f(m-1, n) + c_n f(m-1, n-1)$ , where  $m \leq n$  are two nonnegative integers,  $a_n$  and  $c_n$  are two infinite positive series w.r.t.  $n$ . And  $\forall 0 \leq l \leq n, f(0, l) = 1$ . Then

$$f(m, n) = \sum_{k=1}^m \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} c_n \cdots c_{n-k+1} a_n^{s_0} \cdots a_{n-k}^{s_k} + a_n^m$$

**Proof:** By induction on  $m$ . Obviously, the case  $p=0$  is trivial. Suppose when  $p \leq m-1$   $\forall p \leq l \leq n, f(p, l) = \sum_{k=1}^p \sum_{s_0+s_1+\dots+s_k=p-k} c_l \cdots c_{l-k+1} a_l^{s_0} \cdots a_{l-k}^{s_k} + a_l^p$  holds, then when  $p=m$ , we have

$$\begin{aligned} a_n f(m-1, n) &= \sum_{k=1}^{m-1} \sum_{s_0+s_1+\dots+s_k=m-1-k} c_n \cdots c_{n-k+1} a_n^{s_0+1} \cdots a_{n-k}^{s_k} + a_n^m \\ &= \sum_{k=1}^{m-1} \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0 \geq 1}} c_n \cdots c_{n-k+1} a_n^{s_0} \cdots a_{n-k}^{s_k} + a_n^m \end{aligned}$$

and

$$\begin{aligned} c_n f(m-1, n-1) &= \sum_{k=1}^{m-1} \sum_{s_0+s_1+\dots+s_k=m-1-k} c_n \cdots c_{n-k} a_{n-1}^{s_0} \cdots a_{n-1-k}^{s_k} + c_n a_{n-1}^{m-1} \\ &= \sum_{k=1}^{m-1} \sum_{s_1+s_2+\dots+s_{k+1}=m-1-k} c_n \cdots c_{n-k} a_{n-1}^{s_1} \cdots a_{n-1-k}^{s_{k+1}} + c_n a_{n-1}^{m-1} \\ &= \sum_{k=2}^m \sum_{s_1+s_2+\dots+s_k=m-k} c_n \cdots c_{n-k+1} a_{n-1}^{s_1} \cdots a_{n-k}^{s_k} + c_n a_{n-1}^{m-1} \\ &= \sum_{k=1}^m \sum_{s_1+s_2+\dots+s_k=m-k} c_n \cdots c_{n-k+1} a_{n-1}^{s_1} \cdots a_{n-k}^{s_k} \\ &= \sum_{k=1}^m \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0=0}} c_n \cdots c_{n-k+1} a_n^{s_0} a_{n-1}^{s_1} \cdots a_{n-k}^{s_k} \end{aligned}$$

From the above two equation, there holds



$$\begin{aligned}
f(m, n) &= a_n f(m-1, n) + c_n f(m-1, n-1) \\
&= \sum_{k=1}^m \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} c_n \cdots c_{n-k+1} a_n^{s_0} \cdots a_{n-k}^{s_k} + a_n^m
\end{aligned}$$

The previous  $n$  can be replaced by any  $l$ , where  $m \leq l \leq n$   $\square$

Using lemma1 we can easily obtain two following Theorems.

**Theorem3.** Choose  $A_{m,n}$  u.a.r. from  $\mathcal{A}(m, n)$ ,  $m \leq n$ , or equivalently let  $A_{m,n}$  from  $\mathcal{A}(m, n, 1/2)$ . Then

$$E_{\mathcal{A}}(AM(A_{m,n})) = \sum_{k=0}^m C_m^k \frac{P_n^k}{2^k}$$

where  $C_m^k = \frac{m!}{k!(m-k)!}$  and  $P_n^k = \frac{n!}{(n-k)!}$

**Proof:** Induction on  $m$ . The case  $p=0$ ,  $E_{\mathcal{A}}(AM(A)) = 1$  is trivial. Suppose  $\forall p \leq m-1$ ,

$$p \leq l \leq n \quad E_{\mathcal{A}}(AM(A_{p,l})) = \sum_{k=0}^p C_p^k \frac{P_l^{p-k}}{2^{p-k}} = \sum_{k=0}^p C_p^k \frac{P_l^k}{2^k}$$

when  $p=m$ ,  $\forall m \leq l \leq n$ , we have

$$\begin{aligned}
E_{\mathcal{A}}(AM(A_{m,l})) &= E_{\mathcal{A}}(AM(A_{m,l}^{1,0})) + \sum_{j=1}^n a_{1,j} AM(A_{m,l}^{1,j}) \\
&= E_{\mathcal{A}}(AM(A_{m-1,l})) + \sum_{j=1}^n E_{\mathcal{A}}(a_{1,j}) E_{\mathcal{A}}(AM(A_{m-1,l-1})) \\
&= E_{\mathcal{A}}(AM(A_{m-1,l})) + \frac{n}{2} E_{\mathcal{A}}(AM(A_{m-1,l-1}))
\end{aligned}$$

Using lemma1, here  $a_l \equiv 1$ , and  $c_l = \frac{l}{2}$  then

$$\begin{aligned}
E_{\mathcal{A}}(AM(A_{m,l})) &= \sum_{k=1}^m \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} c_l \cdots c_{l-k+1} + 1 \\
&= \sum_{k=1}^m \frac{P_l^k}{2^k} \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} 1 + 1 \\
&= \sum_{k=1}^m \frac{P_l^k}{2^k} C_m^k + 1 \\
&= \sum_{k=0}^m \frac{P_l^k}{2^k} C_m^k
\end{aligned}$$

□

**Theorem4** Choose  $A_{m,n}$  u.a.r. from  $\mathcal{A}(m,n)$ ,  $m \leq n$ , and let  $X_{A_{m,n}}$  be the output by AMM. Then

$$E_{\mathcal{A}}(E_{\sigma}(X_{A_{m,n}})) = \sum_{k=0}^m C_m^k \frac{P_n^k}{2^k}$$

and

$$E_{\mathcal{A}}(E_{\sigma}(X_{A_{m,n}}^2)) = \sum_{k=0}^m \frac{P_n^k P_{n+3}^k}{2^{m+k}} \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} (n+2)^{s_0} (n+2-1)^{s_1} \dots (n+2-k)^{s_k}$$

**Proof:** The first equation is trivial since  $E_{\sigma}(X_{A_{m,l}}^2) = AM(A_{m,l})$ . For the second one, we use induction on  $m$ . The case  $p=0$  is obvious. Suppose  $\forall A_{p,l}$  where  $0 \leq p \leq m-1$ ,  $p \leq l \leq n$  the second equation holds. When  $p = m$ , noting the fact  $M = |W| - 1$  is a binomial variable with parameter  $l$  and  $1/2$  (recall  $W/\{0\}$  is the set of column indices with a 1 in the first row), then

$$\begin{aligned} E_{\mathcal{A}}(E_{\sigma}(X_{A_{m,l}}^2)) &= \sum_{q=0}^l E_{\mathcal{A}}(E_{\sigma}(X_{A_{m,l}}^2) | M = q) Pr_{\mathcal{A}}(M = q) \\ &= \sum_{q=0}^l E_{\mathcal{A}}((q+1) \sum_{j \in W} E_{\sigma}(X_{A_{m,l}}^2) | M = q) Pr_{\mathcal{A}}(M = q) \\ &= \sum_{q=0}^l E_{\mathcal{A}}((q+1) E_{\sigma}(X_{A_{m-1,l}}^2) + q(q+1) E_{\sigma}(X_{A_{m-1,l-1}}^2)) Pr_{\mathcal{A}}(M = q) \\ &= (E_{\mathcal{A}}(M) + 1) E_{\mathcal{A}}(E_{\sigma}(X_{A_{m-1,l}}^2)) + (E_{\mathcal{A}}(M^2) + E_{\mathcal{A}}(M)) E_{\mathcal{A}}(E_{\sigma}(X_{A_{m-1,l-1}}^2)) \\ &= (\frac{l+2}{2}) E_{\mathcal{A}}(E_{\sigma}(X_{A_{m-1,l}}^2)) + (\frac{l^2+3l}{4}) E_{\mathcal{A}}(E_{\sigma}(X_{A_{m-1,l-1}}^2)) \end{aligned}$$

Using lemma1, here  $a_l = \frac{l+2}{2}$ , and  $c_l = \frac{l^2+3l}{4}$ . Then

$$\begin{aligned} E_{\mathcal{A}}(E_{\sigma}(X_{A_{m,l}}^2)) &= \sum_{k=1}^m \frac{P_l^k P_{l+3}^k}{4^k} \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} (\frac{l+2}{2})^{s_0} (\frac{l+2-1}{2})^{s_1} \dots (\frac{l+2-k}{2})^{s_k} + (\frac{l+2}{2})^m \\ &= \sum_{k=0}^m \frac{P_l^k P_{l+3}^k}{2^{k+m}} \sum_{\substack{s_0+s_1+\dots+s_k=m-k \\ s_0, \dots, s_k \geq 0}} (l+2)^{s_0} (l+2-1)^{s_1} \dots (l+2-k)^{s_k} \end{aligned}$$

□

**Theorem5** Choose  $A_{n,n}$  u.a.r. from  $\mathcal{A}(n,n)$ , and let  $X_{A_{n,n}}$  be the output by AMM. Then **whp**  $h(n) \leq E_{\mathcal{A}}(E_{\sigma}(X_{A_{n,n}})) \leq nh(n)$ , where  $h(n) = \frac{(n!)^2}{2^n} \frac{2^{k^*}}{(n-k^*)!(k^*)^2}$ ,  $k^* = \lfloor -1 + \sqrt{2n+3} \rfloor$ .

where  $\lfloor * \rfloor$  denotes the largest integer no more than  $*$ .

**Proof:**

$$\begin{aligned} E_{\mathcal{A}}(E_{\sigma}(X_{A_{n,n}})) &= \sum_{k=0}^n C_n^k \frac{P_n^k}{2^k} \\ &= \sum_{k=0}^n C_n^k \frac{P_n^{n-k}}{2^{n-k}} \\ &= \frac{(n!)^2}{2^n} \sum_{k=0}^n \frac{2^k}{(n-k)!(k!)^2} \end{aligned}$$

and let  $b_k = \frac{2^k}{(n-k)!(k!)^2}$ , then  $\frac{b_k}{b_{k-1}} = \frac{2(n-k+1)}{k^2}$ , set  $\frac{b_k}{b_{k-1}} \geq 1$  we have  $k \leq -1 + \sqrt{2n+3}$ , thus,

$b_{k^*} = \max_{k=0, \dots, n} b_k$ . Thus, obviously

$$\frac{(n!)^2}{2^n} b_{k^*} \leq E_{\mathcal{A}}(E_{\sigma}(X_{A_{n,n}})) \leq n \frac{(n!)^2}{2^n} b_{k^*}$$

□

**Theorem6** Choose  $A_{n,n}$  u.a.r. from  $\mathcal{A}(n, n)$ , and let  $X_{A_{n,n}}$  be the output by AMM. Then whp

$$\frac{E_{\mathcal{A}}(E_{\sigma}(X_{A_{n,n}}^2))}{E_{\mathcal{A}}^2(E_{\sigma}(X_{A_{n,n}}))} \geq n^{(\sqrt{n}/2)}$$

**Proof:** Numerical experiment shows the above result. however the theoretical analysis seems so hard than until now I haven't thought out the way to show the comparably tight for  $E_{\mathcal{A}}(E_{\sigma}(X_{A_{n,n}}^2))$  since the order of  $\sum_{s_0+s_1+\dots+s_k=n-k} (n+2)^{s_0}(n+2-1)^{s_1} \dots (n+2-k)^{s_k}$  is too difficult to gain a good lower bound. The following bound is easy to check and the best one among methods I thought out,

$$E_{\mathcal{A}}(E_{\sigma}(X_{A_{n,n}}^2)) \geq \sum_{k=0}^n \frac{(n!)^2(n+3)!}{2^{2n}} \frac{2^k(k+2)^k}{(k!)^2(k+3)!(n-k)!}$$

However it still can't reach the goal. Therefore, the proof of this theorem will be left for the future.

Even if Theorem6 has been proved, unfortunately, the critical ratio for almost all the matrices can not obtained from this theorem since two random variables are not independent. In order to accomplish the ultimate result, we need to calculate the  $E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{n,n}}^2))$ . Using the induction similar to theorem4, we can obtain the recursion of  $E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m,n}}^2))$  (recall  $M$  is a binomial variable with parameter  $n$  and  $\frac{1}{2}$ ).

$$\begin{aligned} E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m,n}}^2)) &= 2(E_{\mathcal{A}}(M^3) + 2E_{\mathcal{A}}(M^2) + E_{\mathcal{A}}(M))E_{\mathcal{A}}(E_{\sigma}(X_{A_{m-1,n}}^2)E_{\sigma}(X_{A_{m-1,n-1}}^2)) \\ &+ (E_{\mathcal{A}}(M^2) + 2E_{\mathcal{A}}(M) + 1)E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m-1,n}}^2)) + (E_{\mathcal{A}}(M^4) + 2E_{\mathcal{A}}(M^3) + E_{\mathcal{A}}(M^2))E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m-1,n-1}}^2)) \end{aligned}$$

Comparing  $E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m,n}}^2))$  with  $E_{\mathcal{A}}^2(E_{\sigma}(X_{A_{n,n}}^2))$  and computing their ratio have to be done. Our main aim of doing this is to find the matrices satisfying  $E_{\sigma}(X_{A_{m,n}}^2) \leq E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m,n}}^2))g(n)$ , where  $g(n)$  is a polynomial of  $n$ . However, the ratio of  $\frac{E_{\mathcal{A}}(E_{\sigma}^2(X_{A_{m,n}}^2))}{E_{\mathcal{A}}^2(E_{\sigma}(X_{A_{n,n}}^2))}$  is so large that it can't accomplish our goal. Thus we deduce our requirement **whp** to with a certain probability  $p > 0$ , and in our results  $p = \frac{1}{2} - \varepsilon$  where  $\varepsilon$  is no more than 0.02. To prove the theorem, we need the following lemma, which will be proved in section IV.

**Lemma2** Let  $\mathcal{B}(m, n)$  denote the set of all  $n \times n$  0-1 matrices with exact  $m$  1's,  $m \gg n$ . Choose  $B$  u.a.r. from  $\mathcal{B}(m, n)$ . Then

$$E(AM(B)) = \sum_{k=0}^n (C_n^k)^2 k! \frac{C_{n^2-k}^{m-k}}{C_{n^2}^m}$$

and

$$\frac{E(AM^2(B))}{E^2(AM(B))} = 1 + o(1), n \rightarrow \infty$$

**Theorem7** Choose  $A_{n,n}$  u.a.r. from  $\mathcal{A}(n, n)$ , and let  $X_{A_{n,n}}$  be the output by AMM. Then

$$Pr\left(\frac{E_{\sigma}(X_{A_{n,n}}^2)}{E_{\sigma}^2(X_{A_{n,n}})} \geq n^{\sqrt{n}/2-1}\right) \geq \frac{\sum_{i=(1/2+\varepsilon)n^2}^{n^2} C_{n^2}^k}{2^{n^2}}$$

where  $c$  is a constant no more 10, and  $\varepsilon \leq 0.02$ .

**Proof:** From lemma2 we know if we set  $m = (1/2 + \varepsilon)n^2$  and  $q = \frac{C_{n^2-k}^{m-k}}{C_{n^2}^m}$ . When  $n$  goes to infinity, noting  $k \leq n \ll m, n^2$ , there holds

$$q = \frac{C_{n^2-k}^{m-k}}{C_{n^2}^m} = \frac{m(m-1) \cdots (m-k)}{n^2(n^2-1) \cdots (n^2-k)}$$

and

$$\begin{aligned} \ln(q) &= \sum_{i=0}^{k-1} [\ln(m-i) - \ln(n^2-i)] \\ &= k \ln\left(\frac{m}{n^2}\right) + \sum_{i=0}^{k-1} \left[ \ln\left(1 - \frac{i}{m}\right) - \ln\left(1 - \frac{i}{n^2}\right) \right] \\ &= k \ln\left(\frac{m}{n^2}\right) - \sum_{i=0}^{k-1} \left[ \frac{i}{m} - \frac{i}{n^2} + O\left(\frac{i^2}{m^2}\right) \right] \\ &= k \ln\left(\frac{m}{n^2}\right) - \frac{k(k-1)}{2} \left( \frac{1}{m} - \frac{1}{n^2} \right) + O\left(\frac{k^3}{m^2}\right) \end{aligned}$$

Thus, noting that  $km^{-1} \leq 2nm^{-1} = O(n^3m^{-2})$

$$\begin{aligned} q &= \left(\frac{m}{n^2}\right)^k \exp\left[-\frac{k^2}{2}\left(\frac{1}{m} - \frac{1}{n^2}\right) + O\left(\frac{n^3}{m^2}\right)\right] \\ &= \left(\frac{(1/2 + \varepsilon)n^2}{n^2}\right)^k \exp\left[-\frac{k^2}{2}\left(\frac{1}{(1/2 + \varepsilon)n^2} - \frac{1}{n^2}\right) + O\left(\frac{n^3}{((1/2 + \varepsilon)n^2)^2}\right)\right] \\ &\leq e^{-1}(1/2 + \varepsilon)^k \end{aligned}$$

Let  $B$  selected u.a.r. from  $\mathcal{B}(m, n)$  Since  $\frac{E(AM^2(B))}{E^2(AM(B))} = 1 + o(1)$ , as  $n \rightarrow \infty$  then  $Pr(AM(B) < \frac{5}{6}E(AM(B))) \rightarrow 0$ , as  $n \rightarrow \infty$ . So, if  $m \geq (1/2 + \varepsilon)n^2$  and  $\varepsilon \leq 0.02$ , we have **whp**

$$\begin{aligned} E_\sigma(X_B^2) &\geq E_\sigma^2(X_B) \\ &= AM^2(B) \\ &\geq \left(\frac{5}{6}E(AM(B))\right)^2 \\ &= \left(\frac{5}{6} \sum_{k=0}^n (C_n^k)^2 k! \frac{C_{n^2-k}^{m-k}}{C_{n^2}^m}\right)^2 \\ &\geq \left(\sum_{k=0}^n (C_n^k)^2 k! \frac{5e^{-1}}{6} (1/2 + \varepsilon)^k\right)^2 \\ &\geq \sum_{k=0}^n \frac{P_n^k P_{n+3}^k}{2^{n+k}} \sum_{\substack{s_0+s_1+\dots+s_k=n-k \\ s_0, \dots, s_k \geq 0}} (n+2)^{s_0} (n+2-1)^{s_1} \dots (n+2-k)^{s_k} \\ &= E_{\mathcal{A}}(E_\sigma(X_{A_{n,n}})). \end{aligned}$$

Noting  $Pr(A \in \bigcup_{m \geq (1/2 + \varepsilon)n^2} \mathcal{B}(m, n)) = \frac{\sum_{i=(1/2 + \varepsilon)n^2}^{n^2} C_{n^2}^k}{2^{n^2}},$

thus  $Pr(E_\sigma(X_{A_{n,n}}^2) \geq E_{\mathcal{A}}(E_\sigma(X_{A_{n,n}}))) \geq \frac{\sum_{i=(1/2 + \varepsilon)n^2}^{n^2} C_{n^2}^k}{2^{n^2}}.$

Using Markov's inequality,

$$Pr(E_\sigma(X_{A_{n,n}}^2) \geq nE_{\mathcal{A}}(E_\sigma(X_{A_{n,n}}))) \leq \frac{1}{n} \rightarrow 0$$

then **whp**  $E_\sigma(X_{A_{n,n}}^2) \leq nE_{\mathcal{A}}(E_\sigma(X_{A_{n,n}}))$ . Finally, we have

$$Pr\left(\frac{E_\sigma(X_{A_{n,n}}^2)}{E_\sigma^2(X_{A_{n,n}})} \geq \frac{1}{n} \frac{E_{\mathcal{A}}(E_\sigma(X_{A_{n,n}}^2))}{E_{\mathcal{A}}(E_\sigma(X_{A_{n,n}}))}\right) \geq \frac{\sum_{i=(1/2 + \varepsilon)n^2}^{n^2} C_{n^2}^k}{2^{n^2}}$$

Apply theorem6 to the above formula, we have

$$Pr\left(\frac{E_\sigma(X_{A_{n,n}}^2)}{E_\sigma^2(X_{A_{n,n}})} \geq n^{\sqrt{n}/2-1}\right) \geq \frac{\sum_{i=(1/2+\varepsilon)n^2}^{n^2} C_{n^2}^k}{2^{n^2}}$$

#### IV. THE NUMBER OF ALL THE MATCHINGS ON RANDOM GRAPH.

In this section, we consider the expectation and variance of the number of all the matchings on  $G$  selected u.a.r. from  $\mathcal{G}(m, n)$ . We have the following theorem.

**Theorem8** Choose  $G$  u.a.r. from  $\mathcal{G}(m, n)$ , where  $\mathcal{G}(m, n)$  denotes the set of bipartite graph with  $\#V_1 = \#V_2 = n$  as its vertices and exact  $m$  edges,  $m \gg n$ , and let  $AM(G)$  denotes the number of all the matchings in  $G$ . Then we have

$$E(AM(G)) = \sum_{k=0}^n (C_n^k)^2 k! E(X_{M(k)})$$

and

$$\begin{aligned} E(AM^2(G)) &= \sum_{k=0}^n \sum_{i=0}^k (C_n^k)^2 k! \sum_{p=0}^{\min(i, n-k)} C_{n-k}^p C_k^{i-p} P_{n-i+p}^p \sum_{j=0}^{i-p} C_{i-p}^j [F_{n-j}(i-p-j)] E(X_{M(k+i-j)}) \\ &+ \sum_{k=1}^n \sum_{i=0}^{k-1} (C_n^k)^2 k! \sum_{p=0}^{\min(i, n-k)} C_{n-k}^p C_k^{i-p} P_{n-i+p}^p \sum_{j=0}^{i-p} C_{i-p}^j [F_{n-j}(i-p-j)] E(X_{M(k+i-j)}) \end{aligned}$$

where  $E(X_{M(k)}) = C_{n^2-k}^{m-k}/C_{n^2}^m$  and  $F_n(p) = \sum_{r=0}^p (-1)^r C_p^r P_{n-r}^{p-r}$

**Proof:** we'll use the methodology in [6]; Let  $M(k)$  be a  $k$ -matching on  $V_1 + V_2$ , For  $G \in \mathcal{G}(m, n)$ , define the random variable  $X_M(G)$  to be 1 if  $M(k)$  is contained in  $G$ , and otherwise 0. The expectation and second moment of  $AM(G)$  is as follows.

$$E(AM(G)) = E\left(\sum_{k=0}^n \sum_{M(k)} X_{M(k)}\right) = \sum_{k=0}^n \sum_{M(k)} E(X_{M(k)})$$

and

$$E(AM^2(G)) = E\left(\left(\sum_{k=0}^n \sum_{M(k)} X_{M(k)}\right)^2\right) = \sum_{k=0}^n \sum_{i=0}^n \sum_{M(k), M'(i)} E(X_{M(k)} X'_{M'(i)})$$

where  $\forall 0 \leq k \leq n$ ,  $M(k)$  and  $M'(k)$  range over all  $(C_n^k)^2 k!$   $k$ -matching's on  $V_1 + V_2$ . Note that

$$E(X_{M(k)}) = \frac{C_{n^2-k}^{m-k}}{C_{n^2}^m}$$

The first equation follows quickly. For the second, in order to compute  $E(X_{M(k)}X'_{M(i)})$ , we have to calculate the number of pairs of  $M(k)$  and  $M'(i)$  as a function of the overlap  $j = |M(k) \cap M'(i)|$ . For any fixed  $k$ , suppose  $i \leq k$ , we need to compute the number of the pairs of  $M(k)$  and  $M'(i)$ , where  $i = 0, \dots, k$ , and  $M'(i)$  ranges over all  $(C_n^i)^2 i!$   $i$ -matching's on  $V_1 + V_2$ . The problem can be equivalently stated as follows: There're  $n$  different letters and  $n$  different envelopes. Among these letters, there're exact  $k$  ( $0 \leq k \leq n$ ) labeled letters, each of which has only one 'mother envelope' among envelopes. Different labeled letters have different mother envelopes. We call a  $j$ -fit if there're exact  $j$  labeled letters put into its own mother envelope. Now choose  $i$  ( $0 \leq i \leq k$ ) letters from these  $n$  letters, then put them into  $i$  envelopes, and each letter can only be put into one envelope.  $\forall$  possible  $j$ , how many circumstances of  $j$ -fit are there? We can solve this problem like this: Suppose there're  $p$  letters unlabeled and  $i - p$  labeled letters among the selected letters, obviously,  $0 \leq p \leq \min(n - k, i)$ , the number of ways of choosing letters is  $C_{n-k}^p C_k^{i-p}$ . If the labeled letters has been laid, then the number of the ways of putting  $p$  unlabeled letters is  $P_{n-(i-p)}^p$ . For any  $j$  ( $0 \leq j \leq i - p$ ), there're  $C_{i-p}^j$  ways putting exact  $j$  labeled letters in its own mother envelope. The last one we need to deal with is how many ways to put  $i - p - j$  labeled letters into  $n - j$  envelopes which contain all these  $i - p - j$  letters' mother envelopes, satisfying 0-fit. By the principle of inclusion-exclusion see[1], we can easily obtain the number of the ways is  $F_{n-j}(i - p - j)$ , where  $F_n(p) = \sum_{r=0}^p (-1)^r C_p^r P_{n-r}^{p-r}$ . Noting that  $p$  ranges over 0 to  $\min(i, n - k)$ , and  $j$  ranges over 0 to  $i - p$ , for each  $k$  and  $i \leq k$ . Then

$$\sum_{M'(i)} E(X_{M(k)}X'_{M(i)}) = \sum_{p=0}^{\min(i, n-k)} C_{n-k}^p C_k^{i-p} P_{n-i+p}^p \sum_{j=0}^{i-p} C_{i-p}^j [F_{n-j}(i - p - j)] E(X_{M(k+i-j)})$$

where  $E(X_{M(k)}) = C_{n^2-k}^{m-k}/C_{n^2}^m$  and  $F_n(p) = \sum_{r=0}^p (-1)^r C_p^r P_{n-r}^{p-r}$ .

Consider,

$$\begin{aligned}
\sum_{k=0}^n \sum_{i=0}^n \sum_{M(k), M'(i)} E(X_{M(k)} X'_{M(i)}) &= \left( \sum_{k=0}^n \sum_{i=0}^k + \sum_{k=0}^{n-1} \sum_{i=k+1}^n \right) \sum_{M(k), M'(i)} E(X_{M(k)} X'_{M(i)}) \\
&= \left( \sum_{k=0}^n \sum_{i=0}^k + \sum_{k=0}^{n-1} \sum_{i=k+1}^n \right) \sum_{M(k), M'(i)} E(X_{M(k)} X'_{M(i)}) \\
&= \left( \sum_{k=0}^n \sum_{i=0}^k + \sum_{i=1}^n \sum_{k=0}^{i-1} \right) \sum_{M(k), M'(i)} E(X_{M(k)} X'_{M(i)}) \\
&= \left( \sum_{k=0}^n \sum_{i=0}^k + \sum_{k=1}^n \sum_{i=0}^{k-1} \right) \sum_{M(k), M'(i)} E(X_{M(k)} X'_{M(i)}) \\
&= \left( \sum_{k=0}^n \sum_{i=0}^k + \sum_{k=1}^n \sum_{i=0}^{k-1} \right) \sum_{M(k)} \sum_{M'(i)} E(X_{M(k)} X'_{M(i)}) \\
&= \left( \sum_{k=0}^n (C_n^k)^2 k! \sum_{i=0}^k + \sum_{k=1}^n (C_n^k)^2 k! \sum_{i=0}^{k-1} \right) \sum_{M'(i)} E(X_{M(k)} X'_{M(i)})
\end{aligned}$$

Replace  $\sum_{M'(i)} E(X_{M(k)} X'_{M(i)})$  by

$\sum_{p=0}^{\min(i, n-k)} C_{n-k}^p C_k^{i-p} P_{n-i+p}^p \sum_{j=0}^{i-p} C_{i-p}^j [F_{n-j}(i-p-j)] E(X_{M(k+i-j)})$ , then the second equation is achieved.

□

**Remark:** To complete the proof of theorem7, we also need to know whether the ratio  $\frac{E(AM^2(G))}{E^2(AM(G))}$  goes to 1 as n goes to infinity, adding the condition such as  $m^2 n^{-3} \rightarrow \infty$  as  $n \rightarrow \infty$ . We guess such a result is right, however the calculus seems very difficult. And this result also contributes to the upper bound of critical ratio for almost all the matrices.

### Acknowledgments

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